MATH ANALYSIS IDENTITIES L.A.P.

TRIGONOMETRY

Suggested time: 1 week

BEHAVIORAL OBJECTIVES

1. Prove trigonometric identities
* 2. Supply necessary restrictions for trigonometric identities.

This L.A.P. looks little but it is mean. It utilizes the concepts developed in the previous L.A.P.s.

Equations such as sin 2 θ = 2 sin θ cos θ were called theorems in the W.H.I.P. L.A.P. They are also called identities. An identity is an equation which holds for all variables for which the involved functions are defined.

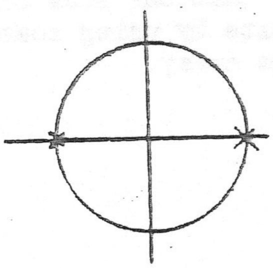
Examples:

(1) $\sin 2\theta = 2 \sin \theta \cos \theta$ for all θ

(2) $\tan \theta = \dfrac{\sin \theta}{\cos \theta}$ for $\theta \neq \dfrac{(2k+1)\pi}{2}$ (For θ equal to an odd multiple of $\frac{\pi}{2}$, cos θ is zero.)

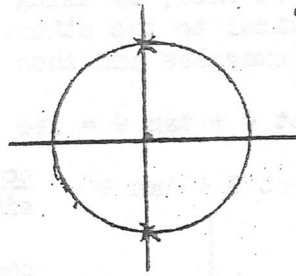
* We have a mathematical responsibility to include the restrictions. In (2) it is important to say $\theta \neq \dfrac{(2k+1)\pi}{2}$, because of the situation of division by zero.

Below is a chart showing the most common restrictions needed.



To restrict the X-intercepts of the unit circle use:

$\theta \neq k\pi$



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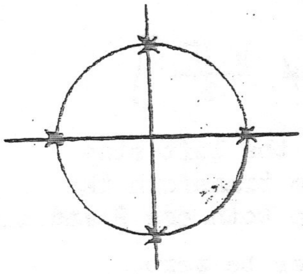


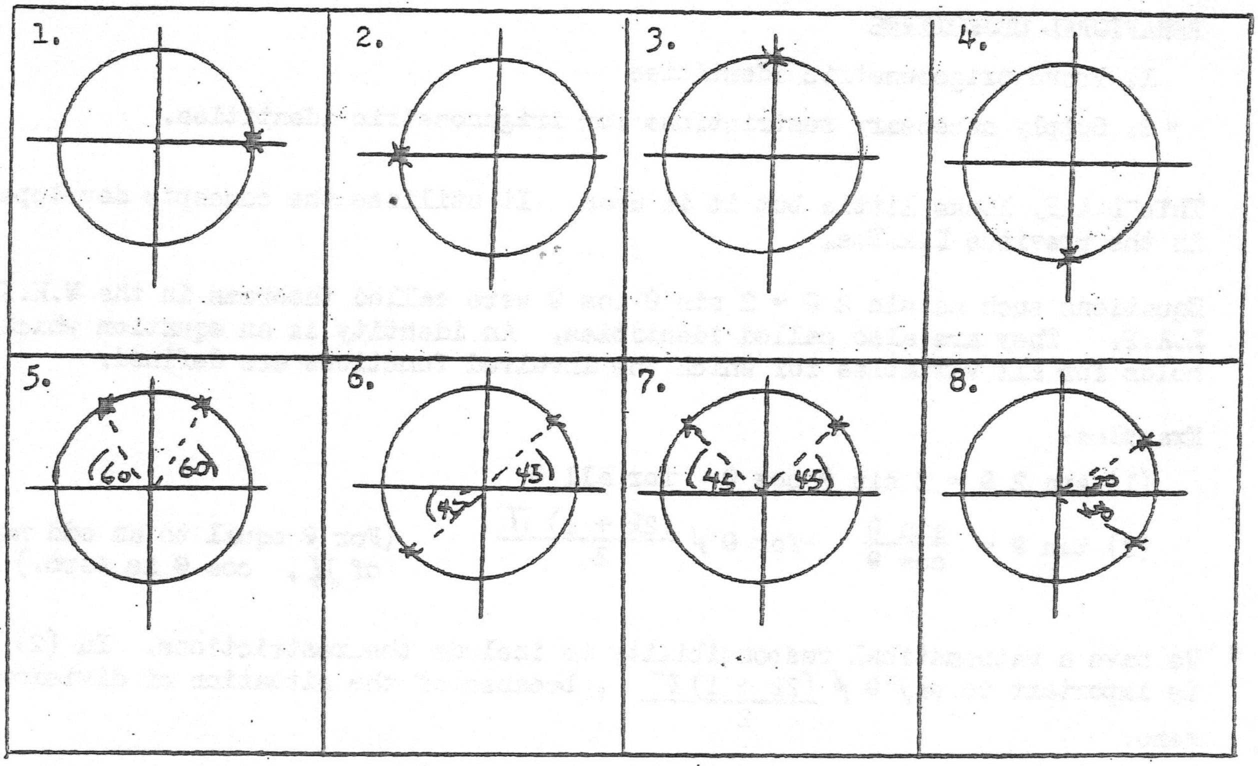
To restrict all the intercepts of the unit circle use:

$\theta \neq \dfrac{k\pi}{2}$

All possible needed restrictions are impossible to demonstrate. At times much ingenuity is needed. The first exercise will give you some practice.

*Optional for Trig students. Trig students should start this L.A.P. on page 3 with Proving Identities.

Exercises #1. (Optional for Trig students)
For each of the following the * indicate restriction spots for θ. Indicate the necessary restriction on θ.



PROVING IDENTITIES

To prove an identity we show, by using the basic trig theorems, that one side of the equation is identical to the other. It is best to demonstrate by using some examples. Study the examples and then practice on your own like crazy.

Example 1: Prove: $\cot\theta + \tan\theta = \sec\theta \csc\theta$

Proof: $\cot\theta + \tan\theta = \dfrac{\cos\theta}{\sin\theta} + \dfrac{\sin\theta}{\cos\theta}$

$$= \frac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos\theta}$$

$$= \frac{1}{\sin\theta\cos\theta}$$

$\cot\theta + \tan\theta = \sec\theta \csc\theta$ $\left[\theta \neq \dfrac{k\pi}{2}\right]$

OBSERVE: Work only on one side of the equation. Here the left side was chosen. Basic trig theorems were used to transform the left side into the right. $\theta \neq \dfrac{k\pi}{2}$ because both $\cos\theta$ and $\sin\theta$ appear as denominators. Neither should ever be zero.

Example 2.

Prove: $\cot\theta - \tan\theta = 2\cot 2\theta$

Proof:

$$2\cot 2\theta = 2\,\frac{\cos 2\theta}{\sin 2\theta}$$

$$= \frac{2(\cos^2\theta - \sin^2\theta)}{2\cos\theta\sin\theta}$$

$$= \frac{\cos\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta}$$

$$2\cot 2\theta = \cot\theta - \tan\theta \qquad \left[\theta \neq \frac{k\pi}{2}\right]$$

Example 3.

Prove: $\dfrac{\cos\theta}{1+\sin\theta} = \dfrac{1-\sin\theta}{\cos\theta}$

Proof:

$$\frac{\cos\theta}{1+\sin\theta} = \frac{\cos\theta\,(1-\sin\theta)}{(1+\sin\theta)(1-\sin\theta)}$$

$$= \frac{\cos\theta\,(1-\sin\theta)}{1-\sin^2\theta}$$

$$= \frac{\cos\theta\,(1-\sin\theta)}{\cos^2\theta}$$

$$\frac{\cos\theta}{1+\sin\theta} = \frac{1-\sin\theta}{\cos\theta} \qquad \left[\theta \neq \frac{(2k+1)\pi}{2}\right]$$

TRICK: We want a $1 - \sin\theta$ on top so we put it there!

To prove an identity:

1. Choose one side, usually the more complicated of the two.
2. Fiddle with the basic trig functions.
3. Make the chosen side look exactly like the other side.

* 4. Provide the necessary restrictions...no zeros in the denominator.

*Optional for trig students.

SOME PRACTICE PROOFS

Do lots of these. After you do a couple show them to your teacher so everybody concerned knows that you are doing them correctly. Be sure to follow the form shown of the previous pages. If you are working for Math Analysis credit also supply the restrictions. The more you practice the better you'll become.

Prove:

1. $\tan\theta \cos\theta = \sin\theta$

2. $(1 - \cos\theta)(1 + \cos\theta) = \sin^2\theta$

3. $\tan\theta = \dfrac{\sin 2\theta}{1 + \cos 2\theta}$

4. $\dfrac{2}{\csc^2\theta} = 1 - \dfrac{1}{\sec 2\theta}$

5. $2\csc 2\theta = \sec\theta \csc\theta$

6. $\tan\theta \sin 2\theta = 2\sin^2\theta$

7. $1 - 2\sin^2\theta + \sin^4\theta = \cos^4\theta$

8. $\dfrac{2\cos^2\theta - \sin^2\theta + 1}{\cos\theta} = 3\cos\theta$

9. $\sin\theta \tan\theta + \cos\theta = \dfrac{1}{\cos\theta}$

10. $\dfrac{1}{\cos^2\theta} + \tan^2\theta + 1 = \dfrac{2}{\cos^2\theta}$

11. $\sin^4\theta - \sin^2\theta\cos^2\theta - 2\cos^4\theta = \sin^2\theta - 2\cos^2\theta$

12. $\sec^2\theta - \csc^2\theta = (\tan\theta + \cot\theta)(\tan\theta - \cot\theta)$

13. $\tan\theta - \tan\phi = \sec\theta \sec\phi \sin(\theta - \phi)$ [Same restriction for both θ & ϕ]

14. $\sin 4\theta = 4\sin\theta\cos\theta\cos 2\theta$

15. $\dfrac{\sin 2\theta}{1 + \cos 2\theta} = \dfrac{1 - \cos 2\theta}{\sin 2\theta}$

16. $\cos 2\theta \cos\theta + \sin 2\theta \sin\theta = \cos\theta$

17. $2\cos^2\dfrac{\theta}{2} - \cos\theta = 1$

18. $(\cos\theta - \sin\theta)^2 = 1 - \sin 2\theta$

19. $4 \sin^2\theta \cos^2\theta = 1 - \cos^2 2\theta$

20. $-\cos^2\theta = \dfrac{\cos^2 2\theta - 1}{4 \sin^2\theta}$

21. $\cos\theta + \sin\theta = \dfrac{\cos 2\theta}{\cos\theta - \sin\theta}$

22. $\tan^2\theta - \sin^2\theta = \dfrac{\sin^4\theta}{\cos^2\theta}$

23. $\cos^2\theta\,(1 - \tan^2\theta) = \cos 2\theta$

24. $\sec^2\theta \cot^2\theta - \cos^2\theta \csc^2\theta = 1$

25. $\dfrac{2 \sin\frac{\theta}{2} \cos\frac{\theta}{2}}{\sin\theta} = \pm 1$

26. $\tan(\theta + \phi) = \dfrac{\cot\phi + \cot\theta}{\cot\theta \cot\phi - 1}$ $\left[\theta + \phi \neq \dfrac{(2k+1)\pi}{2};\ \theta, \phi \neq k\pi\right]$

For more practice try some of the following:

MODERN TRIGONOMETRY, Wooten: Pages 89-90, 1 - 36

TRIGONOMETRY, Drooyan: Page 151, 1-38, Pages 154-5, 1-24.

ELEMENTARY MATH ANALYSIS, Herberg, Pages 347-8, 1 - 42.

(You don't have to check out any books, just select the problems from the book you have.)

Answers: Exercise 1: 1. $\theta \neq 2k$; 2. $\theta \neq (2k + 1)\pi$; 3. $\theta \neq \dfrac{(4k + 1)\pi}{2}$;

4. $\theta \neq \dfrac{(4k + 3)\pi}{2}$; 5. $\theta \neq \dfrac{(4k + 1)\pi}{2} \pm \dfrac{\pi}{6}$; 6. $\theta \neq \dfrac{(4k + 1)\pi}{4}$

7. $\theta \neq \dfrac{(4k + 1)\pi}{2} \pm \dfrac{\pi}{4}$; 8. $\theta \neq 2k\pi \pm \dfrac{\pi}{6}$.

RESTRICTIONS FOR PROOFS:

$\theta \neq \dfrac{(2k + 1)\pi}{2}$ for numbers 1, 3, 6, 8, 9, 10, 13, 22, 23.

$\theta \neq \dfrac{k\pi}{2}$; for numbers 5, 2, 15, 24, $\theta \neq k\pi$; number 20, 25

$\theta \neq k\pi$ and $\theta \neq \dfrac{(2k+1)\pi}{2}$ for number 4

$\theta \neq \dfrac{(4k + 1)\pi}{4}$ for number 21. No restrictions on all other proofs.

There is no trial run for this L.A.P. The Math Analysis test will consist of 8 proofs chosen from 12. It will be necessary to supply restrictions. The Trig test will consist of 6 proofs chosen from 12. No restrictions needed.